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Finitely Generated Free Modular Ortholattices. I

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A description is given of the **n**-generated free algebras in the variety of modular ortholattices generated by an ortholattice $\mathbf{MO}_2$ of height 2 with 4 atoms. In the subvariety lattice of orthomodular lattices, the variety $\mathbf{V}(\mathbf{MO}_2)$ is the unique cover of the variety of Boolean algebras, in which **n**-generated free algebras were described by G. Boole in 1854. It is shown that the **n**-generated free algebra in the variety $\mathbf{V}(\mathbf{MO}_2)$ is a product of the **n** -generated free Boolean algebra $\mathbf{2}^{2^n}$ and $\Phi(n)$ copies of the generator $\mathbf{MO}_2$, and formula is presented for $\Phi(n)$. To achieve this result, algebraic methods of the theory of orthomodular lattices are combined with recently developed methods of natural duality theory for varieties of algebras.

1. ORTHOMODULAR LATTICES

1.1. Introduction

In 1936 G. Birkhoff and J. von Neumann (Birkhoff and von Neumann, 1936) suggested taking the lattice of closed subspaces of a Hilbert space as a suitable model for 'the logic of quantum mechanics.' This lattice equipped with the relation of orthogonal complement can be described as an ortholattice. While in case of a finite-dimensional Hilbert space the ortholattice of its closed subspaces is modular, in the case of an infinite-dimensional Hilbert space the modular law is not satisfied. In 1937 K. Husimi (Husimi, 1937) showed that a weaker law—the so-called orthomodular law—is satisfied in the ortholattice of closed subspaces of any Hilbert space. Since then the theory of orthomodular lattices has been developed; the monographs Kalm-

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bach (1983) and Beran (1984) are highly recommended for the following and other facts about orthomodular lattices.

An *orthomodular lattice* is an algebra $(L; \vee, \wedge, ', 0, 1)$ such that:

(a) $(L; \vee, \wedge)$ is a lattice;
(b) the operation $'$ is order-reversing with respect to the underlying lattice order $\leq$, i.e., $a = a \wedge b$ implies $b' = b' \wedge a'$;
(c) $0 \leq a \leq 1$ for all $a \in L$;
(d) the following laws are satisfied:

$$(a')' = a \tag{1}$$

$$a \wedge a' = 0 \quad \text{and} \quad a \vee a' = 1 \tag{2}$$

$$(a \wedge b)' = a' \vee b' \quad \text{and} \quad (a \vee b)' = a' \wedge b' \tag{3}$$

$$0' = 1, \qquad 1' = 0 \tag{4}$$

$$a \leq b \Rightarrow b = a \vee (b \wedge a') \tag{5}$$

Here (5) is the *orthomodular law.* It has the equivalent form

$$b = (b \wedge a) \vee [b \wedge (b \wedge a)'] \tag{6}$$

Let L be an orthomodular lattice. An important reflexive and symmetric binary relation can be defined on L. This is the *compatibility relation*, referred to as 'a is compatible with b' and denoted $a \leftrightarrow b$, which is defined as follows:

$$a \leftrightarrow b \quad \text{if} \quad a = (a \wedge b) \vee (a \wedge b') \qquad (a, b \in L)$$

It is can be easily verified (Kalmbach, 1983) that the following rules for $\leftrightarrow$ hold in any orthomodular lattice:

$$a \leq b \Rightarrow a \leftrightarrow b \tag{7}$$

$$a \leq b' \Rightarrow a \leftrightarrow b \tag{8}$$

$$a \leftrightarrow b \Rightarrow a \leftrightarrow b', \quad a' \leftrightarrow b, \quad a' \leftrightarrow b' \tag{9}$$

$$a \leftrightarrow b \Leftrightarrow c(a, b) = 1 \tag{10}$$

where

$$c(a, b) = (a \wedge b) \vee (a \wedge b') \vee (a' \wedge b) \vee (a' \wedge b')$$

is the *commutator* of the elements a, b.

Comparing orthomodular lattices with Boolean algebras, one may say that a Boolean algebra is an orthomodular lattice in which every two elements are compatible. On the other hand, in an orthomodular lattice there are also noncompatible pairs in general. Consequently, one cannot use the distributive law in orthomodular lattices as in Boolean algebras. However, the following version of distributivity holds in orthomodular lattices. Let $M \subseteq L$ be such that $\vee M$ exists in L and let $a \in L$ be such that $a \leftrightarrow m$ for every $m \in M$. Then

$$a \leftrightarrow \vee M \quad \text{and} \quad a \wedge (\vee M) = \bigvee_{m \in M} (a \wedge m) \tag{11}$$

For $a \in L$ we let a^i denote a if $i = 0$ and a' if $i = 1$. The commutator of elements $x_1, \ldots, x_n \in L$ is

$$c(x_1, \ldots, x_n) = \bigvee_{(i_1, \ldots, i_n) \in \{0,1\}^n} x_1^{i_1} \wedge \cdots \wedge x_n^{i_n}$$

and we write $c'(x_1, \ldots, x_n)$ for $(c(x_1, \ldots, x_n))'$. We have already presented the commutator of two elements above. The commutator of three elements $x, y, z \in L$ is

$$\begin{aligned} c(x, y, z) = {} & (x \wedge y \wedge z) \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \\ & \vee (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y' \wedge z') \end{aligned}$$

From (7) and (8) it follows that $x_1^{i_1} \wedge \cdots \wedge x_n^{i_n} \leftrightarrow x_i$ for every $i = 1, 2, \ldots, n$ and $i_1, \ldots, i_n \in \{0, 1\}$. Then by (11) we have

$$c(x_1, \ldots, x_n) \leftrightarrow x_i \quad \text{for every} \quad i = 1, 2, \ldots, n \tag{12}$$

For every term function t: $L^n \to L$, the term $t(x_1, \ldots, x_n)$ can, according to (3), be rewritten in a form which uses the operation symbols $\vee$ and $'$ only. Therefore by (9) and (11) we have, for any $a, x_1, \ldots, x_n \in L$,

$$a \leftrightarrow x_i, \qquad i = 1, 2, \ldots, n \Rightarrow a \leftrightarrow t(x_1, \ldots, x_n) \tag{13}$$

1.2. Varieties of Orthomodular Lattices

The basic information about the subvariety lattice of the variety $\mathcal{OM}$ of orthomodular lattices can be found in Kalmbach (1983).

It is known that there is a three-element (covering) chain

$$\mathcal{T} \subsetneq \mathcal{B} \subsetneq \mathcal{MO}_2,$$

(see Fig. 1) at the bottom of the subvariety lattice of $\mathcal{OM}$, where $\mathcal{T}$ and $\mathcal{B}$ are the varieties of trivial algebras and Boolean algebras, respectively, and $\mathcal{MO}_2 = \mathbf{V}(\mathbf{MO}_2)$ is the variety generated by the orthomodular lattice $\mathbf{MO}_2$ of height 2 with 4 atoms a, a', b, b' (see Fig. 2).

In general, $\mathbf{MO}_n$ ($n \geq 2$) is the orthomodular lattice of height 2 with $2n$ atoms $a_1, a_1', \ldots, a_n, a_n'$. Every atom a_i generates a maximal Boolean

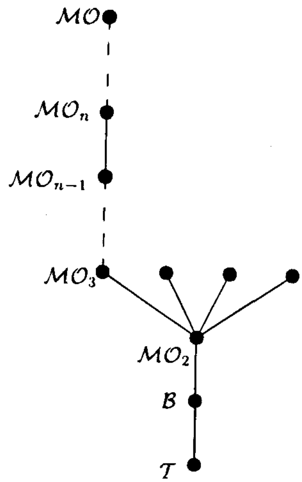

Fig. 1.

subalgebra $\{0, a_i, a_i', 1\}$ of $\mathbf{MO}_n$ which is called a *block* of $\mathbf{MO}_n$. Obviously, each $\mathbf{MO}_n$ $(n \geq 2)$ satisfies the modular law

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$$

The lattices $\mathbf{MO}_n$ $(n \geq 2)$ are the only finite subdirectly irreducible modular ortholattices and they generate a chain of varieties

$$\mathcal{MO}_2 \subsetneq \mathcal{MO}_3 \subsetneq \cdots \subsetneq \mathcal{MO}_{n-1} \subsetneq \mathcal{MO}_n \subsetneq \cdots \subsetneq \mathcal{MO}$$

of type $\omega + 1$, where $\mathcal{MO}_n = \mathbf{V}(\mathbf{MO}_n)$ is the variety generated by $\mathbf{MO}_n$ and

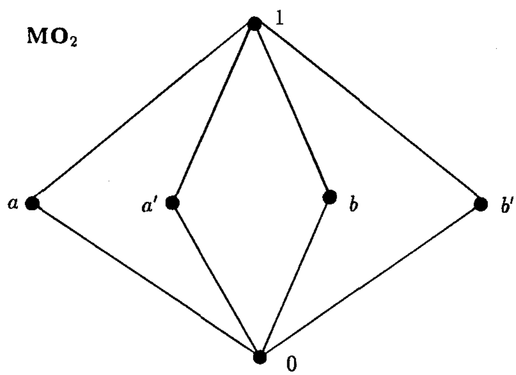

Fig. 2.

$\mathcal{MO}$ is the variety of all modular ortholattices. The strict inclusions $\mathcal{MO}_n \subsetneq \mathcal{MO}_{n+1}$ follow from the fact that the identity

$$\bigwedge_{\substack{i,j=1 \\ i<j}}^{n+1} c'(x_i, x_j) = 0$$

holds in $\mathcal{MO}_n$, but not in $\mathcal{MO}_{n+1}$.

The variety $\mathcal{MO}$ of modular ortholattices is not generated by the finite members, as, for example, the identity

$$c(x, c(y, z)) = 1$$

holds in every $\mathbf{MO}_n$ ($n \geq 2$), but does not hold in the modular ortholattice of all subspaces of a three-dimensional Hilbert space over $\mathbb{R}$ (Bruns, 1976).

1.3. Intervals in Orthomodular Lattices

Intervals in orthomodular lattices are defined in the usual way, as in ordered sets. We shall deal only with intervals of the form $[0, v]$ ($v \in L$); these can be considered to be orthomodular lattices if one defines the orthocomplement of an element $a \in [0, v]$ in $[0, v]$ to be $a' \wedge v$.

An element $a \in L$ is called *central* if it is compatible with every $x \in L$. The set of all central elements of L is said to be the *center* of L and denoted $Z(L)$. The center of L is a subalgebra of L which is a Boolean algebra (Kalmbach, 1983). It is easy to prove that

$$a \in Z(L),\ v \in L \Rightarrow a \wedge v \in Z([0, v]) \tag{14}$$

On the other hand, not every element of $Z([0, v])$ can be written as $a \wedge v$ for some $a \in Z(L)$.

We conclude this subsection with an important result of MacLaren (1964; see also Kalmbach, 1983, p. 20):

$$c \in Z(L) \Leftrightarrow L \cong [0, c] \times [0, c'] \tag{15}$$

1.4. Free Orthomodular Lattices

A free orthomodular lattice with one generator $F_{\mathcal{OM}}(1)$ is obviously isomorphic to the four-element Boolean algebra $\{0, x, x', 1\}$. Thus $F_{\mathcal{OM}}(1) \cong F_{\mathcal{B}}(1) \cong \mathbf{2}^2$ (here $\mathbf{2}$ denotes the two-element Boolean algebra $\mathbf{2} = (\{0, 1\}; \vee, \wedge, ', 0, 1)$).

A free orthomodular lattice with two generators $F_{\mathcal{OM}}(2)$ is a direct product of the free Boolean algebra with two generators $F_{\mathcal{B}}(2)$ and the lattice $\mathbf{MO}_2$. Thus

$$F_{\mathcal{OM}}(2) \cong F_{\mathcal{B}}(2) \times \mathbf{MO}_2 \cong \mathbf{2}^4 \times \mathbf{MO}_2 \cong F_{\mathcal{MO}_2}(2)$$

This free algebra has 96 elements and is described in detail in Beran (1984).

A free orthomodular lattice with three generators $F_{\mathbb{O}\mathcal{M}}(3)$ is infinite. Even the free modular ortholattice $F_{\mathcal{M}\mathbb{O}}(3)$ is infinite, since it has the orthomodular lattice of closed subspaces of $\mathbb{R}^3$ as a homomorphic image (Kalmbach, 1983, p. 229).

In this paper we use methods of natural duality theory (Davey and Werner, 1983; Clark and Davey, 1998) to describe the structure of the free modular ortholattices $F_{\mathcal{M}\mathbb{O}_2}(n)$ with n generators ($n \geq 3$) in the variety $\mathcal{M}\mathbb{O}_2$ that covers the variety of Boolean algebras. The springboard for the work was the calculation of $F_{\mathcal{M}\mathbb{O}_2}(3)$ by the last author. This was done using computer programs developed to determine and optimize natural dualities for quasi-varieties generated by small finite algebras (Priestley and Ward, 1994; Wegener, n.d.).

2. A NATURAL DUALITY FOR THE VARIETY $\mathcal{M}\mathbb{O}_2$

The theory of natural dualities concerns the topological representation of algebras. It grew out of two classical dualities—Pontryagin's duality for abelian groups (Pontryagin, 1966) and Stone's duality for Boolean algebras (Stone, 1936).

An important step toward its development was Priestley's duality for distributive lattices (Priestley, 1970, 1972). A rapid growth of the theory over the last 15 years, which started with a paper of Davey and Werner (1983), has recently led to a monograph (Clark and Davey, 1998), which will likely be the standard reference on the topic.

The main idea of the theory is that, given a quasi-variety $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ of algebras generated by an algebra $\underline{\mathbf{M}}$, one can often find a topological relational structure $\underset{\sim}{\mathbf{M}}$ on the underlying set M of $\underline{\mathbf{M}}$ such that a dual equivalence exists between $\mathcal{A}$ and a suitable category $\mathcal{X}$ of topological relational structures of the same type as $\underset{\sim}{\mathbf{M}}$. Requiring the relational structure of $\underset{\sim}{\mathbf{M}}$ to be 'algebraic over $\underline{\mathbf{M}}$,' all the requisite category theory 'runs smoothly.' A uniform way of representing each algebra $\mathbf{A}$ in the quasi-variety $\mathcal{A}$ as an algebra of continuous structure-preserving maps from a suitable structure $\mathbf{X} \in \mathcal{X}$ into $\underset{\sim}{\mathbf{M}}$ is obtained. In particular, the representation is relatively simple for free algebras in $\mathcal{A}$.

We shall now recall the basic scheme of the theory more precisely; for more detail see Davey and Werner (1983), Davey (1993), or Clark and Davey (1998). Let $\underset{\sim}{\mathbf{M}} = (M; F)$ be a finite algebra. We write this as $\underline{\mathbf{M}}$ where we wish to stress that the set M is carrying the structure of an algebra. Let $\underset{\sim}{\mathbf{M}} = (M; G, H, R, \tau)$ be a discrete topological structure, i.e., the set M endowed with (finite) families G, H, and R of operations, partial operations, and relations, respectively, and with a discrete topology τ. We recall that the graph of an n-ary (partial) operation $g\colon M^n \to M$ is the $(n+1)$-ary relation

$$g^{\square} = \{(x_1, \ldots, x_n, g(x)) \mid (x_1, \ldots, x_n) \in M^n\} \subseteq M^{n+1}$$

We say that the structure $\underset{\sim}{\mathbf{M}}$ is *algebraic over* $\underline{\mathbf{M}}$ if the relations in R and the graphs of the operations and partial operations in $G \cup H$ are subalgebras of appropriate powers of $\underline{\mathbf{M}}$. Note that a unary (partial) operation is algebraic over $\underline{\mathbf{M}}$ if and only if it is a (partial) endomorphism of $\underline{\mathbf{M}}$.

Let $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\underline{\mathbf{M}})$ be the quasi-variety generated by $\underline{\mathbf{M}}$ and assume that $\underset{\sim}{\mathbf{M}} = (M; G, H, R, \tau)$ is algebraic over $\underline{\mathbf{M}}$. Let $\mathcal{M} = \mathbb{I}\mathbb{S}_C\mathbb{P}(\underset{\sim}{\mathbf{M}})$ be the 'topological quasi-variety' generated by $\underset{\sim}{\mathbf{M}}$, i.e., the class of all structures which are embeddable as closed substructures into powers of $\underset{\sim}{\mathbf{M}}$. For any algebra $\mathbf{A} \in \mathcal{A}$, let $D(\mathbf{A})$ denote the set of all $\mathcal{A}$-homomorphisms $\mathbf{A} \to \underline{\mathbf{M}}$. Since $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$, $D(\mathbf{A})$ can naturally be understood as a substructure of $\underset{\sim}{\mathbf{M}}^A$, and so as a member of $\mathcal{X}$.

Let $X \subseteq M^I$ for some set I and let $r \subseteq M^n$ be an n-ary relation on M. We say that a map $\varphi\colon X \to M$ preserves the relation r if $[\varphi(\tilde{x}_1), \ldots, \varphi(\tilde{x}_n)] \in r$ for all $\tilde{x}_1 = (x_{1i})_{i\in I}, \ldots, \tilde{x}_n = (x_{ni})_{i\in I}$ such that $[x_{1i}, \ldots, x_{ni}] \in r$ for every $i \in I$. We say that φ preserves an n-ary (partial) operation if φ preserves its graph as an $(n+1)$-ary relation.

Let $\mathbf{X}$ be a structure in $\mathcal{X}$. By an $\mathcal{X}$-morphism $\varphi\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ we mean a continuous structure-preserving map, i.e., a continuous map preserving all (partial) operations in $G \cup H$ and all relations in R. Let $E(\mathbf{X})$ be the set of all $\mathcal{X}$-morphisms $\mathbf{X} \to \underset{\sim}{\mathbf{M}}$. Again, since $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$, $E(\mathbf{X})$ can be understood as a subalgebra of $\underline{\mathbf{M}}^X$, i.e., a member of $\mathcal{A}$.

The (hom-)functors $D\colon \mathcal{A} \to \mathcal{X}$ and $E\colon \mathcal{X} \to \mathcal{A}$ are contravariant and dually adjoint. Moreover, for any $\mathbf{A} \in \mathcal{A}$ and for any $\mathbf{X} \in \mathcal{X}$, we have maps $e_A\colon \mathbf{A} \to ED(\mathbf{A})$ and $\epsilon_X\colon \mathbf{X} \to DE(\mathbf{X})$ given by evaluation, namely

$$e_A(a)(h) = h(a) \quad \text{for every} \quad a \in A \quad \text{and} \quad h \in D(\mathbf{A})$$

$$\epsilon_X(y)(\varphi) = \varphi(y) \quad \text{for every} \quad y \in X \quad \text{and} \quad \varphi \in E(\mathbf{X})$$

which are embeddings. We say in this situation that $\underset{\sim}{\mathbf{M}}$ *yields a pre-duality on* $\mathcal{A}$.

Let $\underset{\sim}{\mathbf{M}} = (M; G, H, R, \tau)$ be algebraic over $\underline{\mathbf{M}}$, so that $\underset{\sim}{\mathbf{M}}$ yields a pre-duality on $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\underline{\mathbf{M}})$. We say that $\underset{\sim}{\mathbf{M}}$ *yields a (natural) duality on* $\mathcal{A}$ if for every $\mathbf{A} \in \mathcal{A}$ the embedding e_A is an isomorphism, i.e., the evaluation maps $e_A(a)$ $(a \in A)$ are the only $\mathcal{X}$-morphisms from $D(\mathbf{A})$ to $\underset{\sim}{\mathbf{M}}$. Sometimes we say that $G \cup H \cup R$ *yields a (natural) duality on* $\mathcal{A}$. We further say that $\underset{\sim}{\mathbf{M}}$ (or $G \cup H \cup R$) yields a *full* duality on $\mathcal{A}$ if $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ and for every $\mathbf{X} \in \mathcal{X}$ the embedding ϵ_X is also an isomorphism. In such a case the categories $\mathcal{A}$ and $\mathcal{X}$ are dually equivalent via categorical anti-isomorphisms D and E which are inverse to each other.

Again assume $\underset{\sim}{\mathbf{M}} = (M; G, H, R, \tau)$ is algebraic over $\underline{\mathbf{M}}$ and let r be an n-ary algebraic relation on M (i.e., a subalgebra of $\underline{\mathbf{M}}^n$). We say that the structure $\underset{\sim}{\mathbf{M}}$ (or just $G \cup H \cup R$) *entails* r if for every $\mathbf{X} \in \mathcal{X}$, each $\mathcal{X}$-morphism $\varphi: \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ preserves r. The set $G \cup H \cup R$ entails an n-ary (partial) operation h if it entails its graph $h^\square$ as an $(n + 1)$-ary relation. Also, $G \cup H \cup R$ is said to entail a set K if it entails each $k \in K$. If $G \cup H \cup R$ yields a duality on $\mathcal{A}$, then the duality is not destroyed by deleting from $G \cup H \cup R$ any element which is entailed by the remaining members. This is the key to obtaining economical dualities, and thus dualities which are easy to work with. For a full discussion of the central role played by entailment in duality theory see Davey *et al.* (1995a,b). Let us mention here just one easy fact: if e,s are (partial) endomorphisms of an algebra $\underline{\mathbf{M}}$, then $\{e,s\}$ entails the composition $s \circ e$ and the intersection $s^\square \cap e^\square$.

We have not claimed above that it is always possible, for a given algebra $\underline{\mathbf{M}}$, to choose a structure $\underset{\sim}{\mathbf{M}}$ on M yielding a duality on $\mathbb{ISP}(\underline{\mathbf{M}})$. Indeed, there are algebras $\underline{\mathbf{M}}$ which fail to be 'dualizable' (Davey and Werner, 1983, p. 151; Davey, 1993, p. 107). However, for a very wide range of algebras dualities do exist. For example, the NU-Duality Theorem (Davey and Werner, 1983, Theorem 1.18; Davey, 1993, Theorem 2.8) guarantees that a duality on $\mathbb{ISP}(\underline{\mathbf{M}})$ is available whenever $\underline{\mathbf{M}}$ has a lattice reduct, as is the case for an ortholattice. Many further theorems which say how to choose an appropriate structure $\underset{\sim}{\mathbf{M}}$ on M to obtain a duality, or a full duality, on $\mathbb{ISP}(\underline{\mathbf{M}})$ can be found in Clark and Davey (1998). One of these, called the Arithmetic-Strong-Duality Theorem, will be applied in our investigations. We recall that a variety generated by an algebra $\underline{\mathbf{M}}$ is arithmetical if and only if $\underline{\mathbf{M}}$ has an arithmeticity (Pixley) term $p(x, y, z)$: $\underline{\mathbf{M}}^3 \to \underline{\mathbf{M}}$ satisfying

$$p(a, b, b) = p(a, b, a) = p(b, b, a) = a \quad \text{for all} \quad a, b \in M$$

The following result is an immediate consequence of the Arithmetic-Strong-Duality Theorem (Clark and Davey, 1997, Theorem 3.10).

Theorem 2.1. Assume that a subdirectly irreducible algebra $\underline{\mathbf{M}}$ generates an arithmetical variety $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$. Let $\mathcal{P}_1$ be the set of all unary (partial) endomorphisms of $\underline{\mathbf{M}}$. Then any set H of unary (partial) endomorphisms of $\underline{\mathbf{M}}$ that entails $\mathcal{P}_1$ yields a duality on $\mathcal{A}$.

One of the basic facts of natural duality theory is that if $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$, then each χ-morphism from $\underset{\sim}{\mathbf{M}}^S$ to $\underset{\sim}{\mathbf{M}}$ is an S-ary term function (see, for example, Davey, 1993, p. 87). Conversely, every S-ary term function from $\underline{M}^S$ to $\underline{M}$ is an χ-morphism, provided $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{M}$. Since, as is well known, a free S-generated algebra $\underset{\sim}{F}_{\mathcal{A}}(S)$ in the variety $\mathcal{A}$ generated by $\underline{M}$ is isomorphic to the algebra of all term functions from $\underline{M}^S$ to $\underline{M}$ (with projections as free generators), we immediately get:

Theorem 2.2. Let $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\underline{M})$ be a variety and let $\underset{\sim}{\mathbf{M}} = (M; G, H, R, \tau)$ yield a duality on $\mathcal{A}$. The free algebra $F_{\mathcal{A}}(S)$ generated by a set S in the variety $\mathcal{A}$ is isomorphic to the algebra of all χ-morphisms from $\underset{\sim}{\mathbf{M}}^S$ to $\underset{\sim}{\mathbf{M}}$.

In particular, if S is finite, with $|S| = n$, then the n-generated free algebra $F_{\mathcal{A}}(n) = F_{\mathcal{A}}(S)$ in the variety generated by $\underline{M}$ is isomorphic to the algebra of all $(G \cup H \cup R)$-preserving functions from M^n to M.

The variety $\mathcal{B}$ of Boolean algebras is $\mathbb{I}\mathbb{S}\mathbb{P}(\underline{M})$, where $\underline{M}$ is the two-element Boolean algebra $\mathbf{2} = (\{0, 1\}; \vee, \wedge, ', 0, 1)$. The Stone duality (Stone, 1936) says that $\underset{\sim}{\mathbf{M}} = (\{0, 1\}; \tau)$ (an empty relational structure) yields a duality on $\mathcal{B}$. Hence every Boolean algebra $\mathbf{B}$ is isomorphic to the algebra of all continuous maps from $D(\mathbf{B})$ to $\underset{\sim}{\mathbf{M}}$ and the n-generated free Boolean algebra $F_{\mathcal{B}}(n)$ is isomorphic to the algebra of all functions from 2^n to 2 $(2 = \{0, 1\})$, i.e.,

$$F_{\mathcal{B}}(n) \cong \mathbf{2}^{2^n}$$

We shall focus on the variety $\mathcal{MO}_2 = \mathbf{V}(\mathbf{MO}_2)$ which covers the variety $\mathcal{B}$ in the subvariety lattice of orthomodular lattices. The variety $\mathcal{MO}_2$ is arithmetical since one can define an arithmeticity (Pixley) term on the generator $\mathbf{MO}_2$, for example, as follows:

$$\begin{aligned} p(x, y, z) = {} & (x \vee z) \wedge (x \vee y') \wedge (z \vee y') \\ & \wedge [(c(x, y) \wedge z) \vee (c(y, z) \wedge x) \vee (c(x, z) \wedge x \wedge z)] \end{aligned}$$

For verification of this fact note the following: if x, z belong to the same block of $\mathbf{MO}_2$, then $(x \vee z) \wedge (x' \vee z) = z$ and $c(x, z) = 1$; if x, z are atoms of different blocks of $\mathbf{MO}_2$, $(x \vee z) \wedge (x' \vee z) = 1$ and $c(x, z) = 0$. Thus, by Theorem 2.1, each subset H of the set $\mathcal{P}_1$ of all unary (partial) endomorphisms of $\mathbf{MO}_2$ such that H entails $\mathcal{P}_1$ yields a duality on $\mathcal{MO}_2$. We shall show that the required set H of (partial) endomorphisms can be chosen such that it consists only of the automorphisms s, t of $\mathbf{MO}_2$ (see Fig. 2 again) with graphs

$$s^{\square} = \{(0, 0), (a, a), (a', a'), (b, b'), (b', b), (1, 1)\} \tag{16}$$

$$t^{\square} = \{(0, 0), (a, b), (a', b'), (b, a), (b', a'), (1, 1)\} \tag{17}$$

and one unary partial endomorphism r with the graph

$$r^{\square} = \{(0, 0), (a, 0), (a', 1), (1, 1)\} \tag{18}$$

Indeed, it is easy to verify that the set $\mathcal{P}_1$ on $\mathbf{MO}_2$ consists only of eight automorphisms and five partial endomorphisms and that each of them, except the last one, can be obtained from $\{r, s, t\}$ by composition. These maps are as follows:

Automorphisms of $\mathbf{MO}_2$ (presented as products of disjoint cycles):

$$
\begin{array}{ll}
(b\ b') = s, & (a\ b')(a'\ b) = s \circ t \circ s \\
(a\ b)(a'\ b') = t, & (a\ a') = t \circ s \circ t \\
(a\ b'\ a'\ b) = s \circ t, & (a\ a')(b\ b') = s \circ t \circ s \circ t \\
(a\ b\ a'\ b') = t \circ s, & \mathrm{id} = s \circ s
\end{array}
$$

Partial endomorphisms of $\mathbf{MO}_2$ (presented by their graphs):

$$
\begin{array}{l}
\{(0, 0), (a, 0), (a', 1), (1, 1)\} = r^{\square} \\
\{(0, 0), (b, 0), (b', 1), (1, 1)\} = (r \circ t)^{\square} \\
\{(0, 0), (b, 1), (b', 0), (1, 1)\} = (r \circ t \circ s)^{\square} \\
\{(0, 0), (a, 1), (a', 0), (1, 1)\} = (r \circ t \circ s \circ t)^{\square} \\
\{(0, 0), (1, 1)\} = s^{\square} \cap t^{\square}
\end{array}
$$

Hence $\{r, s, t\}$ entails $\mathcal{P}_1$. We have proved the following result.

Theorem 2.3. $H = \{r, s, t\}$ yields a duality on $\mathcal{MO}_2 = \mathbb{ISP}(\mathbf{MO}_2)$.

From Theorems 2.2 and 2.3 we immediately get:

Corollary 2.4. The n-generated free algebra $F_{\mathcal{MO}_2}(n)$ in the variety $\mathcal{MO}_2$ is isomorphic to the algebra of all $\{r, s, t\}$-preserving functions from $(MO_2)^n$ to MO_2.

3. FINITELY GENERATED FREE ALGEBRAS IN $\mathcal{MO}_2$

Let $F_{\mathcal{MO}_2}(n)$ be the free orthomodular lattice with n generators in the variety $\mathcal{MO}_2 = \mathbb{ISP}(\mathbf{MO}_2)$. As we have already mentioned, $F_{\mathcal{MO}_2}(n)$ is isomorphic to the algebra of all n-ary term functions $t(x_1, \ldots, x_n)$: $(\mathbf{MO}_2)^n \to \mathbf{MO}_2$.

We claim that a term function $t(x_1, \ldots, x_n)$: $(\mathbf{MO}_2)^n \to \mathbf{MO}_2$ which only takes the values from the center $Z(\mathbf{MO}_2) = \{0, 1\}$ is a central element of $F_{\mathcal{MO}_2}(n)$. Indeed, it suffices to show that, for any function $u(x_1, \ldots, x_n) \in F_{\mathcal{MO}_2}(n)$,

$$
\begin{aligned}
t(x_1, \ldots, x_n) = {} & [t(x_1, \ldots, x_n) \wedge u(x_1, \ldots, x_n)] \\
& \vee [t(x_1, \ldots, x_n) \wedge u'(x_1, \ldots, x_n)]
\end{aligned}
$$

Since $t(x_1, \ldots, x_n) \in \{0, 1\}$, equality trivially holds.

It is easy to see that the commutator function $c(x_1, \ldots, x_n)$: $(\mathbf{MO}_2)^n \to \mathbf{MO}_2$ takes only the values 0 and 1. Moreover, one can easily verify that

$$
c'(x_1, \ldots, x_n) = 1 \quad \text{if and only if at least two of } x_1, \ldots, x_n \text{ are atoms of different blocks in } \mathbf{MO}_2 \tag{19}
$$

We have shown that the commutator function $c(x_1, \ldots, x_n)$ is a central element of $F_{\mathcal{MO}_2}(n)$. Thus, by (15),

$$F_{\mathcal{MO}_2}(n) \cong [0, c(x_1, \ldots, x_n)] \times [0, c'(x_1, \ldots, x_n)]$$

It is well known (Kalmbach, 1983, p. 231) that $[0, c(x_1, \ldots, x_n)]$ is a Boolean algebra, the free Boolean algebra with n generators. Hence the problem of describing the structure of $F_{\mathcal{MO}_2}(n)$ reduces to the problem of describing the structure of the interval

$$[0, c'(x_1, \ldots, x_n)]$$

For $n = 1$, $c'(x_1) = 0$. Thus, we shall concentrate on $n \geq 2$.

The Case $n = 2$. The interval $[0, c'(x, y)]$ in the free algebra $F_{\mathcal{MO}_2}(2)$ with two generators consists of elements of the form $t(x, y) \wedge c'(x, y)$, where $t(x, y) \in F_{\mathcal{MO}_2}(2)$. Using (3), we can rewrite the term $t(x, y)$ in a form $l(x, y, x', y')$, where $l(z_1, \ldots, z_4)$ is a lattice term in which x, y, x', y' are substituted for $z_1, \ldots, z_4$. Since by (9), (12), and (13), $c'(x, y)$ is compatible with every element of the free algebra $F_{\mathcal{MO}_2}(2)$, using (11) we get

$$\begin{aligned} t(x, y) \wedge c'(x, y) &= l(x, y, x', y') \wedge c'(x, y) \\ &= l(x \wedge c'(x, y), y \wedge c'(x, y), x' \wedge c'(x, y), y' \wedge c'(x, y)) \end{aligned}$$

Thus, the interval $[0, c'(x, y)]$ in $F_{\mathcal{MO}_2}(2)$ is generated by

$$x \wedge c'(x, y), \quad y \wedge c'(x, y), \quad x' \wedge c'(x, y), \quad \text{and} \quad y' \wedge c'(x, y)$$

It is easy to compute that these four term functions generate the orthomodular lattice

$$\{0, x \wedge c'(x, y), x' \wedge c'(x, y), y \wedge c'(x, y), y' \wedge c'(x, y), c'(x, y)\}$$

isomorphic to $\mathbf{MO}_2$ (Fig. 3). Hence

$$F_{\mathcal{MO}_2}(2) \cong [0, c(x, y)] \times [0, c'(x, y)] \cong F_{\mathcal{B}}(2) \times \mathbf{MO}_2$$

which is a known result from Kalmbach (1983, p. 239) and Beran (1984, Section III.2).

The Case $n = 3$. According to (15), we can decompose the interval $[0, c'(x, y, z)]$ of $F_{\mathcal{MO}_2}(3)$ by the central elements $c(x, y)$, $c(x, z)$, and $c(y, z)$. We get

$$[0, c'(x, y, z)] \cong [0, C_1(x, y, z)] \times [0, C_2(x, y, z)] \times \cdots \times [0, C_8(x, y, z)] \tag{20}$$

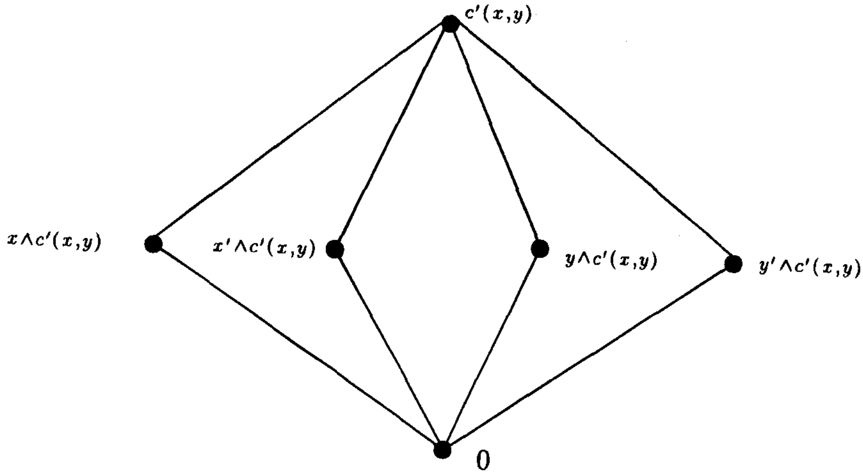

Fig. 3.

where

$$C_1(x, y, z) = c(x, y) \wedge c(x, z) \wedge c(y, z) \wedge c'(x, y, z)$$

$$C_2(x, y, z) = c(x, y) \wedge c(x, z) \wedge c'(y, z) \wedge c'(x, y, z)$$

$$C_3(x, y, z) = c(x, y) \wedge c'(x, z) \wedge c(y, z) \wedge c'(x, y, z)$$

$$C_4(x, y, z) = c'(x, y) \wedge c(x, z) \wedge c(y, z) \wedge c'(x, y, z)$$

$$C_5(x, y, z) = c(x, y) \wedge c'(x, z) \wedge c'(y, z) \wedge c'(x, y, z)$$

$$C_6(x, y, z) = c'(x, y) \wedge c(x, z) \wedge c'(y, z) \wedge c'(x, y, z)$$

$$C_7(x, y, z) = c'(x, y) \wedge c'(x, z) \wedge c(y, z) \wedge c'(x, y, z)$$

$$C_8(x, y, z) = c'(x, y) \wedge c'(x, z) \wedge c'(y, z) \wedge c'(x, y, z)$$

It is easy to see that the functions $C_1(x, y, z)$, $C_8(x, y, z)$: $(MO_2)^3 \to MO_2$ are identically equal to 0; thus we can cancel the intervals $[0, C_1(x, y, z)]$ and $[0, C_8(x, y, z)]$ in (20). Further, by symmetry, the following sets of intervals have the same structure:

$$[0, C_2(x, y, z)], \quad [0, C_3(x, y, z)], \quad \text{and} \quad [0, C_4(x, y, z)]$$

$$[0, C_5(x, y, z)], \quad [0, C_6(x, y, z)], \quad \text{and} \quad [0, C_7(x, y, z)]$$

Now let us take the interval $[0, C_2(x, y, z)]$ and consider the function

$$x \wedge C_2(x, y, z): \quad (MO_2)^3 \to MO_2$$

Note that $C_2(x, y, z) \in \{0, 1\}$. Further, $c'(y, z) = 1$ only if y, z are atoms of different blocks in $\mathbf{MO}_2$; if this holds, then $c(x, y) \wedge c(x, z) = 1$ only if $x \in Z(\mathbf{MO}_2) = \{0, 1\}$. Hence the function $x \wedge C_2(x, y, z)$ takes only the values 0 and 1, and thus it is a central element of $F_{\mathcal{MO}_2}(3)$. By (14) it is also a central element of the interval $[0, C_2(x, y, z)]$, which therefore can, according to (15), be decomposed as

$$[0, C_2(x, y, z)] \cong [0, x \wedge C_2(x, y, z)] \times [0, x' \wedge C_2(x, y, z)]$$

We have

$$x \wedge C_2(x, y, z) = 1 \text{ iff}$$

$$x = 1 \text{ and } y, z \text{ are atoms of different blocks in } \mathbf{MO}_2 \tag{21}$$

and

$$x' \wedge C_2(x, y, z) = 1 \text{ iff}$$

$$x = 0 \text{ and } y, z \text{ are atoms of different blocks in } \mathbf{MO}_2 \tag{22}$$

We shall show that the interval $[0, x \wedge C_2(x, y, z)]$ is isomorphic to $\mathbf{MO}_2$. It follows from Corollary 2.4 that this interval consists of all $\{r, s, t\}$-preserving functions $f(x, y, z)$: $(MO_2)^3 \to MO_2$ such that

$$f(x, y, z) \leq x \wedge C_2(x, y, z) \qquad \text{for all} \quad x, y, z \in MO_2$$

Obviously, taking account of (21), it suffices to consider only triples $(x, y, z) \in (MO_2)^3$ where the function $x \wedge C_2(x, y, z)$ takes a nonzero value, i.e., takes the value 1. So let

$$T := \{(x, y, z) \in (MO_2)^3 | x \wedge C_2(x, y, z) = 1\}$$

Certainly T is invariant under the action of the automorphism group

$$\mathrm{Aut}(\mathbf{MO}_2) = \{\mathrm{id}, s, t, s \circ t, t \circ s, s \circ t \circ s, t \circ s \circ t, s \circ t \circ s \circ t\}$$

There are eight elements in T, and these can be expressed as

$$(1, a, b) = (x_0, y_0, z_0)$$

$$(1, a, b') = (s(x_0), s(y_0), s(z_0))$$

$$(1, b, a) = (t(x_0), t(y_0), t(z_0))$$

$$(1, b', a) = ((s \circ t)(x_0), (s \circ t)(y_0), (s \circ t)(z_0))$$

$$(1, b, a') = ((t \circ s)(x_0), (t \circ s)(y_0), (t \circ s)(z_0))$$

$$(1, b', a') = ((s \circ t \circ s)(x_0), (s \circ t \circ s)(y_0), (s \circ t \circ s)(z_0))$$

$$(1, a', b) = ((t \circ s \circ t)(x_0), (t \circ s \circ t)(y_0), (t \circ s \circ t)(z_0))$$

$$(1, a', b') = ((s \circ t \circ s \circ t)(x_0), (s \circ t \circ s \circ t)(y_0), (s \circ t \circ s \circ t)(z_0))$$

(That is, the automorphism group acts transitively on T.)

Let $f(x, y, z)$: $(MO_2)^3 \to MO_2$ be any $\{r, s, t\}$-preserving function such that we have $f(x, y, z) \leq x \wedge C_2(x, y, z)$. Let $f(x_0, y_0, z_0) = c \in MO_2$. Then, necessarily, f preserves all compositions formed from s and t, so

$$f(\alpha(x_0), \alpha(y_0), \alpha(z_0)) = \alpha f(x_0, y_0, z_0) = \alpha(c) \qquad \text{for all} \quad \alpha \in \mathrm{Aut}(\mathbf{MO}_2) \tag{23}$$

Hence f can take any of the six values $c \in MO_2$ on the triple (x_0, y_0, z_0) and its values on the remaining seven triples in T are then prescribed by (23). Note that the restricion $f \restriction T$ of f to T is r-preserving vacuously because $(r(x), r(y), r(z)) \notin T$ for any $(x, y, z) \in T$. Thus there are six $\{r, s, t\}$-preserving functions $f(x, y, z)$ in the interval $[0, x \wedge C_2(x, y, z)]$ and this interval is isomorphic to $\mathbf{MO}_2$.

In the same way, the interval $[0, x' \wedge C_2(x, y, z)] \cong \mathbf{MO}_2$. Hence

$$[0, C_2(x, y, z)] \cong (\mathbf{MO}_2)^2$$

Analogously,

$$[0, C_3(x, y, z)] \cong (\mathbf{MO}_2)^2 \qquad \text{and} \qquad [0, C_4(x, y, z)] \cong (\mathbf{MO}_2)^2$$

Now take the interval $[0, C_5(x, y, z)]$ in $F_{\mathcal{MO}_2}(3)$. Consider the function

$$v(x, y, z) = ((x' \wedge y) \vee (x \wedge y')) \wedge C_5(x, y, z): \quad (MO_2)^3 \to MO_2$$

Since $c'(x, z) \wedge c'(y, z)$ belongs to $\{0, 1\}$ and takes the value 1 if and only if x, y are atoms of one block of $\mathbf{MO}_2$ and z is an atom of the other block, the function $v(x, y, z)$ takes only the values 0 and 1. Hence it is a central element of the interval $[0, C_5(x, y, z)]$, which can therefore be decomposed as

$$\begin{aligned}[0, C_5(x, y, z)] \cong\ & [0, ((x' \wedge y) \vee (x \wedge y')) \wedge C_5(x, y, z)] \\ & \times [0, ((x' \wedge y') \vee (x \wedge y)) \wedge C_5(x, y, z)]\end{aligned}$$

We have

$$((x' \wedge y) \vee (x \wedge y')) \wedge C_5(x, y, z) = 1 \qquad \text{iff } x, y \text{ are different}$$
$$\text{atoms of one block in } \mathbf{MO}_2 \text{ and } z \text{ is an atom of the other block} \tag{24}$$

$$((x' \wedge y') \vee (x \wedge y)) \wedge C_5(x, y, z) = 1 \qquad \text{iff } x = y \text{ is an atom of}$$
$$\text{one block in } \mathbf{MO}_2 \text{ and } z \text{ is an atom of the other block} \tag{25}$$

By (24) the set

$$T^1 := \{(x, y, z) \in (MO_2)^3 | v(x, y, z) = 1\}$$

consists of the following eight triples [the automorphism group Aut($\mathbf{MO}_2$) acts transitively on T']:

$$(a, a', b) = (x_0, y_0, z_0), (s(x_0), s(y_0), s(z_0)),$$
$$(t(x_0), t(y_0), t(z_0)), \ldots,$$
$$((s \circ t \circ s \circ t)(x_0), (s \circ t \circ s \circ t)(y_0), (s \circ t \circ s \circ t)(z_0))$$

Take $f(x, y, z)$: $(MO_2)^3 \to MO_2$ to be any $\{r, s, t\}$-preserving function such that $f(x, y, z) \leq v(x, y, z)$. Note again that $f \restriction T'$ is r-preserving vacuously. If $f(x_0, y_0, z_0) = c \in \mathbf{MO}_2$, then (23) holds for the remaining seven triples. Since c can be any element of MO_2,

$$[0, v(x, y, z)] \cong \mathbf{MO}_2$$

Similarly,

$$[0, ((x' \wedge y') \vee (x \wedge y)) \wedge C_5(x, y, z)] \cong \mathbf{MO}_2$$

whence

$$[0, C_5(x, y, z)] \cong (\mathbf{MO}_2)^2$$

Hence also

$$[0, C_6(x, y, z)] \cong (\mathbf{MO}_2)^2 \quad \text{and} \quad [0, C_7(x, y, z)] \cong (\mathbf{MO}_2)^2$$

Thus, by (20),

$$[0, c'(x, y, z)] \cong (\mathbf{MO}_2)^{12}$$

Consequently,

$$F_{\mathcal{MO}_2}(3) \cong F_{\mathcal{B}}(3) \times (\mathbf{MO}_2)^{12}$$

and

$$|F_{\mathcal{MO}_2}(3)| = 2^8 \cdot 6^{12}$$

The General Case. For n ($n \geq 2$) generators, we decompose the interval $[0, c'(x_1, \ldots, x_n)]$ by the central elements $c(x_i, x_j)$, $i, j = 1, \ldots, n$, $i < j$. We obtain that

$$[0, c'(x_1, \ldots, x_n)] \cong \prod_{\tilde{w} \in \{0,1\}^N} [0, \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \ldots, x_n)]$$

where the product is taken over all N-tuples

$$\tilde{w} = (w_{1,2}, \ldots, w_{1,n}, w_{2,3}, \ldots, w_{n-1,n}) \in \{0, 1\}^N$$

where $N = \binom{n}{2}$ and

$$c^{w_{i,j}}(x_i, x_j) = \begin{cases} c(x_i, x_j), & \text{if} \quad w_{i,j} = 0 \\ c'(x_i, x_j), & \text{if} \quad w_{i,j} = 1 \end{cases}$$

Every term function

$$t_{\tilde{w}}(x_1, \ldots, x_n) = \bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \ldots, x_n)$$

obviously corresponds to the following undirected graph without multiple edges and loops $G^n_{\tilde{w}}$: the vertex set of $G^n_{\tilde{w}}$ is $\{x_1, \ldots, x_n\}$ and x_ix_j is an edge of $G^n_{\tilde{w}}$ if and only if $w_{i,j} = 1$; for example, for $n = 5$ and $\tilde{w} = (0, 1, 1, 0, 1, 1, 0, 0, 0, 0)$, the term function

$$\begin{aligned} t_{\tilde{w}}(x_1, \ldots, x_5) = {} & c(x_1, x_2) \wedge c'(x_1, x_3) \wedge c'(x_1, x_4) \wedge c(x_1, x_5) \wedge c'(x_2, x_3) \\ & \wedge c'(x_2, x_4) \wedge c(x_2, x_5) \wedge c(x_3, x_4) \\ & \wedge c(x_3, x_5) \wedge c(x_4, x_5) \wedge c'(x_1, \ldots, x_5) \end{aligned}$$

corresponds to the graph $G^n_{\tilde{w}}$ in Fig. 4. The term function $t_{\tilde{w}}(x_1, \ldots, x_n)$ corresponding to a graph $G = G^n_{\tilde{w}}$ will be denoted by $C_G(x_1, \ldots, x_n)$.

As we have seen in the case $n = 3$, some of the term functions

$$C_G(x_1, \ldots, x_n) \colon (\mathbf{MO}_2)^n \to \mathbf{MO}_2$$

can be identically equal to 0. We shall now characterize those graphs G for which the associated functions $C_G(x_1, \ldots, x_n)$ are nonzero.

Proposition 3.1. Let $C_G(x_1, \ldots, x_n) \colon (\mathbf{MO}_2)^n \to \mathbf{MO}_2$ be a term function

$$\bigwedge_{\substack{i,j=1 \\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \ldots, x_n)$$

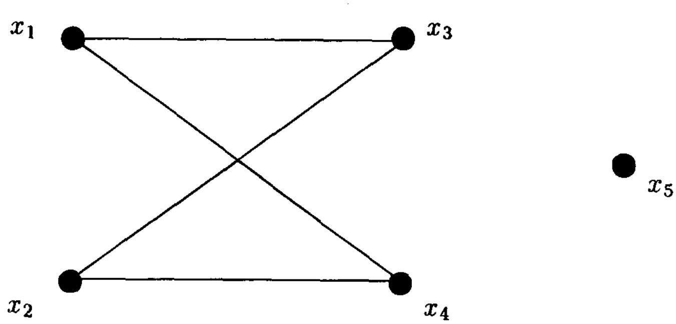

Fig. 4.

with an associated graph $G = G^n_w$. The following conditions are equivalent:

(a) $C_G(x_1, \ldots, x_n)$ is not identically equal to zero;
(b) there exist elements $a_1, \ldots, a_n \in MO_2$, not all from $Z(\mathbf{MO}_2) = \{0,1\}$, with the following properties:
(i) $C_G(a_1, \ldots, a_n) = 1$;
(ii) x_ix_j is an edge of G if and only if a_i, a_j are atoms of different blocks in $\mathbf{MO}_2$;
(c) G consists of k isolated vertices $(0 \le k < n)$ and one connected component which is a complete bipartite graph.

Moreover, provided the equivalent conditions (a)–(c) hold, there are exactly 2^{n+1} n-tuples $(a_1, \ldots, a_n) \in (\mathrm{MO}_2)^n$ with the properties in (b).

Proof. (a) $\Rightarrow$ (b). Assume that

$$C_G(x_1, \ldots, x_n) = \bigwedge_{\substack{i,j=1\\ i<j}}^{n} c^{w_{i,j}}(x_i, x_j) \wedge c'(x_1, \ldots, x_n)$$

is not identically equal to 0. This means that there exist $a_1, \ldots, a_n \in MO_2$ such that $C_G(a_1, \ldots, a_n) \neq 0$, i.e., $C_G(a_1, \ldots, a_n) = 1$, because all $c^{w_{i,j}}(x_i, x_j)$ and $c'x_1, \ldots, x_n)$ take only values 0 and 1 on MO_2. Hence (i) holds. It is easy to see that not all of $a_1, \ldots, a_n$ belong to $Z(\mathbf{MO}_2)$, for otherwise $c'(a_1, \ldots, a_n) = 0$ by (19), and thus $C_G(a_1, \ldots, a_n) = 0$, a contradiction.

From (i) it follows that $c^{w_{i,j}}(a_i, a_j) = 1$ for all $1 \le i < j \le n$. Now if x_ix_j is an edge of G, then $w_{i,j} = 1$, and thus $c'(a_i, a_j) = 1$ yields that a_i, a_j are atoms of different blocks in $\mathbf{MO}_2$ by (19). Conversely, if a_i, a_j are atoms of different blocks in $\mathbf{MO}_2$, then $c'(a_i, a_j) = 1$ by (19), hence $w_{i,j} = 1$ and x_ix_j is an edge of G. We have proved (ii).

(b) $\Rightarrow$ (c). Let us take $(a_1, \ldots, a_n) \in (MO_2)^n$ as in (b). Without loss of generality we may assume that $a_1, \ldots, a_k \in \{0,1\} \subseteq MO_2$, $a_{k+1}, \ldots, a_m$ are atoms of one block of $\mathbf{MO}_2$ and $a_{m+1}, \ldots, a_n$ are atoms of the other block of $\mathbf{MO}_2$ $(0 \le k < m < n)$. By (b)(ii) we conclude that in G each of the vertices $x_1, \ldots, x_k$ is isolated, each of the vertices $x_{k+1}, \ldots, x_m$ is connected with each of the vertices $x_{m+1}, \ldots, x_n$, and no two vertices inside the sets $\{x_{k+1}, \ldots, x_m\}$, $\{x_{m+1}, \ldots, x_n\}$ are connected. Hence G consists of k isolated vertices $x_1, \ldots, x_k$ $(0 \le k < n)$ and one connected component $\{x_{k+1}, \ldots, x_n\}$ which is a complete bipartite graph with partition $\{x_{k+1}, \ldots, x_m\}$, $\{x_{m+1}, \ldots, x_n\}$.

(c) $\Rightarrow$ (a). Let (c) hold and assume without loss of generality that $x_1, \ldots, x_k$ $(0 \le k < n)$ are isolated vertices of G, and let the partition $\{x_{k+1}, \ldots, x_m\}$, $\{x_{m+1}, \ldots, x_n\}$ induce a complete bipartite graph $(k < m < n)$. Let $a_1, \ldots, a_n$ be elements of MO_2 such that $a_1, \ldots, a_k \in \{0,1\}$, $a_{k+1}, \ldots,$

a_m are atoms of one block of $\mathbf{MO}_2$ and $a_{m+1}, \ldots, a_n$ are atoms of the other block of $\mathbf{MO}_2$. For every $i, j \in \{1, \ldots, n\}$ $i < j$, we have the following. If $x_i x_j$ is an edge of G, then $w_{i,j} = 1$ and a_i, a_j are atoms of different blocks of $\mathbf{MO}_2$ and thus $c^{w_{i,j}}(x_i, x_j) = c'(x_i, x_j)$ in $C_G(x_1, \ldots, x_n)$ and $c'(a_i, a_j) = 1$. On the other hand, if $x_i x_j$ is not an edge in G, then $w_{i,j} = 0$ and a_i, a_j are not atoms of different blocks of $\mathbf{MO}_2$ and thus $c^{w_{i,j}}(x_i, x_j) = c(x_i, x_j)$ in $C_G(x_1, \ldots, x_n)$ and $c(a_i, a_j) = 1$. Consequently, $C_G(a_1, \ldots, a_n) = 1$, which proves (a).

From above it follows that if $G = G_w^n$ has k isolated vertices $x_1, \ldots, x_k$ $(0 \le k < n)$ and one connected component with the partition $\{x_{k+1}, \ldots, x_m\}$, $\{x_{m+1}, \ldots, x_n\}$ $(k < m < n)$, then $C_G(a_1, \ldots, a_n) = 1$ on those n-tuples $(a_1, \ldots, a_n) \in (MO_2)^n$ for which $a_1, \ldots, a_k \in Z(\mathbf{MO}_2) = \{0, 1\}$, $a_{k+1}, \ldots, a_m$ are atoms of one block of $\mathbf{MO}_2$, and $a_{m+1}, \ldots, a_n$ are atoms of the other block of $\mathbf{MO}_2$. Each of $a_1, \ldots, a_k$ can be either 0 or 1, i.e., there are 2^k possibilities of choosing $a_1, \ldots, a_k$ in MO_2. Since $\mathbf{MO}_2$ has 2 blocks, there are 2 possibilities of choosing the block from which $a_{k+1}, \ldots, a_m$ are taken (then $a_{m+1}, \ldots, a_n$ are taken from the other block of $\mathbf{MO}_2$). Once it is decided from which block of $\mathbf{MO}_2$ each a_i $(k < i \le n)$ is taken as an atom, there are 2 possibilities in choosing each of these a_i (because both blocks have 2 atoms). Hence there are $2^k \cdot 2 \cdot 2^{n-k} = 2^{n+1}$ possibilities in choosing the elements $a_1, \ldots, a_n$ from MO_2 such that (i) and (ii) of (b) hold. ∎

Let us recall that by Corollary 2.4 the n-generated free algebra $F_{\mathcal{MO}_2}(n)$ is isomorphic to the algebra of all $\{r, s, t\}$-preserving functions from $(MO_2)^n$ to MO_2, where r, s, t are (partial) endomorphisms of $\mathbf{MO}_2$ with graphs $r^\square$, $s^\square$, $t^\square$ given by (16)–(18). Further, a function $f\colon (MO_2)^n \to MO_2$ is r-preserving if and only if, for $a_i, b_i \in MO_2$ $(1 \le i \le n)$,

$$(a_1, b_1) \in r^\square, \ldots, (a_n, b_n) \in r^\square \tag{26}$$

implies

$$(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in r^\square$$

We shall show that, for any graph G satisfying (c) of Proposition 3.1, among the 2^{n+1} n-tuples of $(MO_2)^n$ satisfying (b) of Proposition 3.1, there are no two n-tuples $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in (MO_2)^n$ for which (26) holds. Suppose that such two n-tuples $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ exist. According to (b)(ii) of Proposition 3.1, a_i is a central element of $\mathbf{MO}_2$ if and only if x_i is an isolated vertex of G, which is so if and only if b_i is a central element of $\mathbf{MO}_2$ $(1 \le i \le n)$. Since $r^\square = \{(0, 0), (a, 0), (a', 0), (1, 1)\} \subseteq (MO_2)^2$, (26) automatically yields that for every $i = 1, \ldots, n$ either $a_i = b_i \in Z(\mathbf{MO}_2) = \{0, 1\}$ or a_i is an atom and $b_i \in \{0, 1\}$. Since b_i is central if and only if a_i is central, the latter is impossible. Hence $a_i = b_i \in Z(\mathbf{MO}_2)$ for all $i = 1, \ldots, n$, whence $x_1, \ldots, x_n$ are isolated vertices of G according to (b)(ii) of Proposition 3.1, but this contradicts the fact that G has $k < n$ isolated vertices.

We have characterized those graphs G which correspond to the nontrivial intervals $[0, C_G(x_1, \ldots, x_n)]$ of $F_{\mathcal{MO}_2}(n)$ and we have shown that each nonzero term function $C_G(x_1, \ldots, x_n)$ takes on $(MO_2)^n$ only values 0 and 1, the latter on 2^{n+1} n-tuples from $(MO_2)^n$ among which no two n-tuples satisfy (26). Now we are ready to describe the structure of the intervals $[0, C_G(x_1, \ldots, x_n)]$ in $F_{\mathcal{MO}_2}(n)$.

Let $[0, C_G(x_1, \ldots, x_n)]$ be a nontrivial interval in $F_{\mathbf{MO}_2}(n)$. This interval consists of all $\{r, s, t\}$-preserving functions $f\colon (MO_2)^n \to MO_2$ such that

$$f(a_1, \ldots, a_n) \leq C_G(a_1, \ldots, a_n) \tag{27}$$

for all $(a_1, \ldots, a_n) \in (MO_2)^n$. Let T_G be the 2^{n+1}-element set consisting of the n-tuples $(a_1, \ldots, a_n) \in (MO_2)^n$ for which $C_G(a_1, \ldots, a_n) \neq 0$, i.e., $C_G(a_1, \ldots, a_n) = 1$. It obviously suffices to take account of (27) only for those n-tuples $(a_1, \ldots, a_n)$ which belong to T_G. Since no two elements $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ of T_G satisfy (26), every considered function $f\colon T_G \to MO_2$ is automatically r-preserving. Since the automorphisms s, t generate $\mathrm{Aut}(\mathbf{MO}_2)$, a function $f\colon T_G \to MO_2$ is $\{s, t\}$-preserving if and only if it is $\mathrm{Aut}(\mathbf{MO}_2)$-preserving, i.e., for any automorphism $\alpha \in \mathrm{Aut}(\mathbf{MO}_2)$ and $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in T_G$,

$$(a_1, b_1) \in \alpha^{\square}, \ldots, (a_n, b_n) \in \alpha^{\square} \tag{28}$$

implies

$$(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \alpha^{\square}$$

or, alternatively expressed,

$$f(\alpha(a_1), \ldots, \alpha(a_n)) = \alpha f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n) \tag{29}$$

Let us define a binary relation E on T_G as follows: $((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \in E$ if there is an $\alpha \in \mathrm{Aut}(\mathbf{MO}_2)$ such that (28) holds. It is an easy exercise to show that E is an equivalence on T_G. Each equivalence class $[(a_1, \ldots, a_n)]_E$ is obviously the 8-element set

$$[(a_1, \ldots, a_n)]_E = \{(\alpha(a_1), \ldots, \alpha(a_n)) | \alpha \in \mathrm{Aut}(\mathbf{MO}_2)\}$$

Let $f(x_1, \ldots, x_n)$ be an element of the interval $[0, C_G(x_1, \ldots, x_n)]$ in $F_{\mathcal{MO}_2}(n)$, and thus $f(x_1, \ldots, x_n)\colon (MO_2)^n \to MO_2$ is an $\mathrm{Aut}(\mathbf{MO}_2)$-preserving function such that (27) holds for all $(a_1, \ldots, a_n) \in T_G$. Choose any $(a_1, \ldots, a_n) \in T_G$ and assume that $f(a_1, \ldots, a_n) = c \in MO_2$.

Since (28) holds exactly for the n-tuples $(b_1, \ldots, b_n) \in [(a_1, \ldots, a_n)]_E$, by (29) we get that [similarly to (23) in the case $n = 3$]

$$f(\alpha(a_1), \ldots, \alpha(a_n)) = \alpha f(a_1, \ldots, a_n) = \alpha(c) \tag{30}$$

for all eight maps $\alpha \in \mathrm{Aut}(\mathbf{MO}_2)$. This means that $f\upharpoonright[(a_1, \ldots, a_n)]_E$ can take any of the six values $c \in MO_2$ on the n-tuple $(a_1, \ldots, a_n)$ and its values on the remaining seven n-tuples from $[(a_1, \ldots, a_n)]_E$ are given by (30). Consequently, for given $(a_1, \ldots, a_n) \in T_G$,

$$\{f\upharpoonright[(a_1, \ldots, a_n)]_E | f: T_G \to MO_2 \text{ preserves } \mathrm{Aut}(\mathbf{MO}_2)\} \cong \mathbf{MO}_2$$

Since it is clear that T_G has $2^{n+1}/8 = 2^{n-2}$ equivalence classes $[(a_1, \ldots, a_n)]_E$ [where $(a_1, \ldots, a_n) \in T_G$], the structure of the interval $[0, C_G(x_1, \ldots, x_n)]$ in $F_{\mathcal{MO}_2}(n)$ can be described by

$$\begin{aligned}[0, C_G(x_1, \ldots, x_n)] &\cong \{f\upharpoonright T_G | f: T_G \to MO_2 \text{ preserves } \mathrm{Aut}(\mathbf{MO}_2)\} \\ &\cong (\mathbf{MO}_2)^{2^{n-2}} \end{aligned} \tag{31}$$

Hence we have shown the following result.

Theorem 3.2. For every graph G satisfying (c) in Proposition 3.1, the corresponding nontrivial interval $[0, C_G(x_1, \ldots, x_n)]$ in $F_{\mathcal{MO}_2}(n)$ is isomorphic to $(\mathbf{MO}_2)^{2^{n-2}}$.

To get a complete description of the free algebra $F_{\mathcal{MO}_2}(n)$, it now suffices to count the number of the graphs satisfying (c) of Proposition 3.1. Every such graph $G = G^n_w$ has k isolated vertices $(0 \leq k < n)$. It cannot have $n - 1$ isolated vertices, as this would yield that also the remaining vertex is isolated. Hence $0 \leq k \leq n - 2$. One can choose k isolated vertices by $\binom{n}{k}$ choices. The remaining $(n - k)$ vertices can form a complete bipartite graph by

$$\frac{1}{2}\left(\binom{n-k}{1} + \binom{n-k}{2} + \cdots + \binom{n-k}{n-k-1}\right)$$

possibilities [here $(\binom{n-k}{i} + \binom{n-k}{n-k-i})/2$ means that the bipartite graph, which has $n - k$ vertices divided into two vertex sets, has i vertices in one vertex set and $n - k - i$ in the other].

Hence the number of graphs $G = G^n_w$ associated to the nontrivial intervals $[0, C_G(x_1, \ldots, x_n)]$ of $F_{\mathcal{MO}_2}(n)$ is

$$\begin{aligned}\Phi'(n) &= \sum_{k=0}^{n-2}\binom{n}{k}\cdot\frac{1}{2}\left(\binom{n-k}{1} + \binom{n-k}{2} + \cdots + \binom{n-k}{n-k-1}\right) \\ &= \sum_{k=0}^{n-2}\binom{n}{k}\cdot\frac{(1+1)^{n-k} - 2}{2} \\ &= \frac{1}{2}\cdot\left[\sum_{k=0}^{n-2}\binom{n}{k}\cdot 2^{n-k} - \sum_{k=0}^{n-2}\binom{n}{k}\cdot 2\right]\end{aligned}$$

$$= \frac{1}{2} \cdot [(1 + 2)^n - 2n - 1 - 2 \cdot (2^n - n - 1)]$$

$$= \frac{1}{2} \cdot [3^n - 2^{n+1} + 1]$$

Since

$$F_{\mathcal{MO}_2}(n) \cong F_{\mathcal{B}}(n) \times [0, c'(x_1, \ldots, x_n)]$$

and

$$[0, c'(x_1, \ldots, x_n)] \cong \prod_G [0, C_G(x_1, \ldots, x_n)]$$

where each $[0, C_G(x_1, \ldots, x_n)] \cong (\mathbf{MO}_2)^{2^{n-2}}$, we get that $[0, c'(x_1, \ldots, x_n)]$ is isomorphic to the product of $\Phi(n)$ copies of the generator $\mathbf{MO}_2$ where the formula $\Phi(n)$ is given by

$$\Phi(n) = \frac{1}{2} \cdot (3^n - 2^{n+1} + 1) \cdot 2^{n-2} = 2^{n-3} \cdot (3^n - 2^{n+l} + 1)$$

Hence we get the final results:

Theorem 3.3. For any $n \geq 1$,

$$F_{\mathcal{MO}_2}(n) \cong F_{\mathcal{B}}(n) \times (\mathbf{MO}_2)^{\Phi(n)}$$

where

$$\Phi(n) = 2^{n-3} \cdot (3^n - 2^{n+1} + 1)$$

Corollary 3.4. For any $n \geq 1$,

$$|F_{\mathcal{MO}_2}(n)| = 2^{2^n} \cdot 6^{2^{n-3} \cdot (3^n - 2^{n+1} + 1)}$$

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